Macroscopic quantum tunneling effect of Z_2 topological order

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In this paper, macroscopic quantum tunneling (MQT) effect of Z_2 topological order in the Wen-Plaquette model is studied. This kind of MQT is characterized by quantum tunneling processes of different virtual quasiparticles moving around a torus. By a high-order degenerate perturbation approach, the effective pseudospin models of the degenerate ground states are obtained. From these models, we get the energy splitting of the ground states, which are consistent with those obtained from exact diagonalization method.

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I. INTRODUCTION

In quantum mechanics, quantum tunneling effect is a process during which quantum particles penetrate barriers, which are forbidden in classical processes.[1] It is Gamov who pointed out that a single α particle can tunnel through a potential barrier introducing "macroscopic quantum tunneling" (MQT) into physics for the first time. Macroscopic quantum tunneling effects have been widely applied to different research fields such as quantum oscillations between two degenerate wells of NH_3, quantum coherence in one-dimensional charge-density waves, macroscopic quantum tunneling effect in ferromagnetic single-domain magnets and quantum tunneling phenomena in biased Josephson junctions. In general, to find MQT in a system, there must exist two or more separated "classical" states macroscopically distinct. As shown in Fig. 1, a quantum particle may take a short cut from one well to the other without climbing the barrier.

In this paper we will study a new class of MQT—the MQT in Z_2 topological order. At the beginning we give a brief introduction to Z_2 topological order. Topological order is a new type of quantum order beyond Landau's symmetry-breaking paradigm,[2–4] which exhibits four universal properties: (1) all excitations have mass gaps; (2) the quantum degeneracy of the ground states depends on the genus of the manifold of the background; (3) there are (closed) string net condensations; (4) quasiparticles have exotic statistics. All these properties are robust against local perturbations. Z_2 topological order is the simplest topologically ordered state with three types of quasiparticles: Z_2 charge, Z_2 vortex, and fermions.[5] Z_2 charge and Z_2 vortex are all bosons with mutual π statistics between them. The fermions can be regarded as bound states of a Z_2 charge and a Z_2 vortex. In the last ten years, several exactly solvable spin models with Z_2 topological orders were found such as the Kitaev toric-code model,[6] the Wen-plaquette model,[5,7] and the Kitaev model on honeycomb lattice.[8]

It is known that for Z_2 topological orders on a torus, there always exist four degenerate ground states with the same energy in thermodynamic limit (the so-called topological degeneracy). The different ground states cannot mix into each other through any local fluctuations. However, in a finite system, the degeneracy of the ground states can be (partially) removed via quantum tunneling processes, during which virtual quasiparticles move around the torus.[2,6,9–12] Take the energy splitting from the tunneling process of Z_2 vortex, for example, at first a pair of Z_2 vortices is created; then one of the Z_2 vortices propagates all the way around the torus and annihilates with the other Z_2 vortex.

A decade ago, Kitaev pointed out that the degenerate ground states of a Z_2 topological order make up a protected code subspace (the so-called toric code) free from error.[6,8] In Ref. 9, topological qubit based on the degenerate ground states of a Z_2 topological order has been designed. Then one can manipulate the degenerate ground states of spin models by braiding anyons, which has becomes a hot issue recently.[13–16] Recently, an alternative approach to design topological quantum computation (TQC) is proposed by manipulating the protected code subspace.[11,12] The key point to manipulate the degenerate ground states is to tune their MQT effect. Thus it becomes an interesting issue to study the MQT in Z_2 topological order.

In this paper, by using a high-order degenerate perturbative approach, we study the MQT of the degenerate ground states of Z_2 topological order, taking the Wen-plaquette Model as an example. The remainder of the paper is organized as follows. In Sec. II, the degenerate ground states of the Wen-plaquette model is classified by topological closed-string operators. In Sec. III, the dynamics of quasiparticles are studied. In Sec. IV, the MQT of the degenerate ground states of the Z_2 topological order are formalized on a torus with different lattices. The numerical results are given to compare with the theoretical results. Finally, the conclusions are given in Sec. V.

II. DEGENERATE GROUND STATES AND THEIR REPRESENTATION OF STRING OPERATORS

In this section, we study the degenerate ground states of the Wen-plaquette model. The Hamiltonian of the Wen-plaquette model is given by

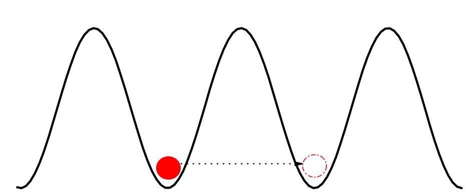

FIG. 1. (Color online) The scheme of a typical macroscopic quantum tunneling process.

$$\hat{H} = -g\sum_i \hat{F}_i, \tag{1}$$

with

$$\hat{F}_i = \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y \tag{2}$$

and $g>0$. σ_i^x and σ_i^y are Pauli matrices on site i. The ground states of the Wen-plaquette model denoted by $F_i \equiv +1$ at each site are known to be an example of Z_2 topological state. The ground-state energy becomes $E_0 = -gN$, where N is the number of the lattice.[2,5,7,10]

In the topological order of the Wen-plaquette model, there exist three types of open-string operators $W_c(C)$, $W_v(C)$ and $W_f(C)$ corresponding to three types of quasiparticles: Z_2 charge, Z_2 vortex, and fermion, respectively.[3] Here C is an open loop. To create a Z_2-vortex (charge) excitation, one may draw a string state that connects nearest neighboring *even* (*odd*) plaquettes $W_v(C)$ [or $W_c(C)$]. Such a string state is created by the following string operator: $\Pi_C \sigma_i^{a_i}$, where the product Π_C is over all the sites on the string along a loop C connecting even plaquettes (or odd plaquettes), $a_i = y$ if i is even and $a_i = x$ if i is odd. For a fermionic excitation, the string operator has a form as $W_f(C) = \Pi_n \sigma_{i_n}^{l_n}$ with a string C connecting the midpoints of the neighboring links, and i_n are sites on the string. $l_m = z$ if the string does not turn at site i_m. $l_m = x$ or y if the string makes a turn at site i_m. $l_m = y$ if the turn forms a upper-right or lower-left corner. $l_m = x$ if the turn forms a lower-right or upper-left corner. It is obvious that the fermionic string can be regarded as a bound state of strings of the Z_2 charges and the Z_2 vortices, which is $W_f(C) = W_c(C)W_v(C)$. If C are closed loops, we get condensed closed-string operators of the ground states $|\Psi_0\rangle$ as

$$\langle\Psi_0|W_c(C)|\Psi_0\rangle = 1, \quad \langle\Psi_0|W_v(C)|\Psi_0\rangle = 1,$$

$$\langle\Psi_0|W_f(C)|\Psi_0\rangle = 1. \tag{3}$$

One can see the detailed definition of the string operators in Ref. 3.

To classify the degeneracy of the ground states on an $L_x \times L_y$ lattice with periodic boundary condition (L_x and L_y are positive integer numbers), we define three types of topological closed-string operators $W_c(\mathcal{C})$, $W_v(\mathcal{C})$, and $W_f(\mathcal{C}) = W_c(\mathcal{C})W_v(\mathcal{C})$, with $\mathcal{C}$ denoting topological closed loops. The word "*topological*" means that the "*big*" loops $\mathcal{C}$ surround the torus globally (see Fig. 2). One can easily check the commutation relations between the topological closed-string operators and the Hamiltonian

$$[H, W_c(\mathcal{C})] = [H, W_v(\mathcal{C})] = [H, W_f(\mathcal{C})] = 0. \tag{4}$$

For the ground states on a torus of an even-by-even ($e*e$) lattice, we can define four types of elementary topological closed-string operators, $W_v(\mathcal{C}_X)$, $W_v(\mathcal{C}_Y)$, $W_f(\mathcal{C}_X)$, and $W_f(\mathcal{C}_Y)$. Here $\mathcal{C}_X$ denotes a closed loop around the torus along e_x direction and $\mathcal{C}_Y$ denotes a closed loop around the torus along e_y direction. Due to the commutation (or anticommutation) relations between them

$$[W_v(\mathcal{C}_X), W_f(\mathcal{C}_X)] = 0, \quad [W_v(\mathcal{C}_Y), W_f(\mathcal{C}_Y)] = 0,$$

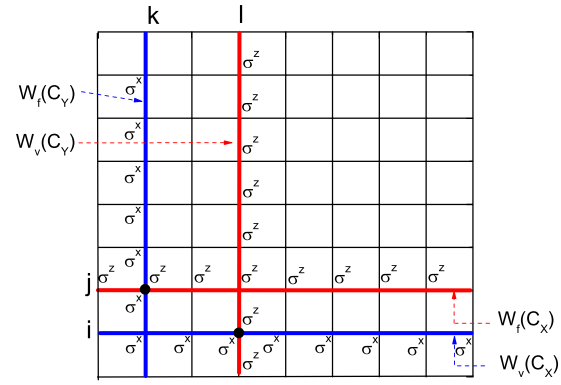

FIG. 2. (Color online) The topological closed-string operators on a torus. The dots denote the crosses of different types of strings.

$$[W_v(\mathcal{C}_X), W_v(\mathcal{C}_Y)] = 0, \quad [W_f(\mathcal{C}_X), W_f(\mathcal{C}_Y)] = 0,$$

$$\{W_v(\mathcal{C}_X), W_f(\mathcal{C}_Y)\} = 0, \quad \{W_v(\mathcal{C}_Y), W_f(\mathcal{C}_X)\} = 0, \tag{5}$$

we may identify $W_v(\mathcal{C}_X)$, $W_v(\mathcal{C}_Y)$, $W_f(\mathcal{C}_X)$, and $W_f(\mathcal{C}_Y)$ by pseudospin operators τ_1^x, τ_2^x τ_2^z, and τ_1^z as

$$W_v(\mathcal{C}_X) = \tau_1^x \otimes \mathbf{1}, \quad W_v(\mathcal{C}_Y) \to \mathbf{1} \otimes \tau_2^x,$$

$$W_f(\mathcal{C}_X) \to \mathbf{1} \otimes \tau_2^z, \quad W_f(\mathcal{C}_Y) \to \tau_1^z \otimes \mathbf{1}. \tag{6}$$

Thus other five topological closed-string operators $W_c(\mathcal{C}_X)$, $W_c(\mathcal{C}_Y)$, $W_c(\mathcal{C}_{XY})$, $W_v(\mathcal{C}_{XY})$, and $W_f(\mathcal{C}_{XY})$ are denoted by $\tau_1^x \otimes \tau_2^z$, $\tau_1^z \otimes \tau_2^x$, $\tau_1^y \otimes \tau_2^y$, $\tau_1^x \otimes \tau_2^x$, and $\tau_1^z \otimes \tau_2^z$, respectively,

$$W_c(\mathcal{C}_X) \to \tau_1^x \otimes \tau_2^z, \quad W_c(\mathcal{C}_Y) \to \tau_1^z \otimes \tau_2^x,$$

$$W_c(\mathcal{C}_{XY}) \to \tau_1^y \otimes \tau_2^y, \quad W_v(\mathcal{C}_{XY}) \to \tau_1^x \otimes \tau_2^x,$$

$$W_f(\mathcal{C}_{XY}) \to \tau_1^z \otimes \tau_2^z. \tag{7}$$

Here $\mathcal{C}_{XY}$ is a closed loop around the torus along diagonal directions. In Table I, the pseudospin representation of the topological closed-string operators are illustrated.

Then as the eigenstates of τ_l^z $(l=1,2)$, the four degenerate ground states are denoted by $|m_1, m_2\rangle = |m_1\rangle \otimes |m_2\rangle$. For $m_l = 0$, we have

$$\tau_l^z |m_l\rangle = |m_l\rangle, \tag{8}$$

and for $m_l = 1$ we have

$$\tau_l^z |m_l\rangle = -|m_l\rangle. \tag{9}$$

Physically, the topological degeneracy arises from the presence or the absence of a π flux of fermion through the hole.

TABLE I. Pseudospin representation of the topological closed-string operators on an even-by-even lattice.

Pseudospin operators	$\mathcal{C}_X$	$\mathcal{C}_Y$	$\mathcal{C}_{XY}$
Z_2 vortex	$\tau_1^x \otimes \mathbf{1}$	$\mathbf{1} \otimes \tau_2^x$	$\tau_1^x \otimes \tau_2^x$
Z_2 charge	$\tau_1^x \otimes \tau_2^z$	$\tau_1^z \otimes \tau_2^x$	$\tau_1^y \otimes \tau_2^y$
Fermion	$\mathbf{1} \otimes \tau_2^z$	$\tau_1^z \otimes \mathbf{1}$	$\tau_1^z \otimes \tau_2^z$

TABLE II. Pseudospin representation of the topological closed-string operators on an even-by-odd lattice.

Pseudospin operators	$\mathcal{C}_X$	$\mathcal{C}_Y$	$\mathcal{C}_{XY}$
Z_2 vortex	τ_1^x		τ_1^x
Z_2 charge	τ_1^x		τ_1^x
Fermion	1	τ_1^z	τ_1^z

The values of m_l reflect the presence ($m_l=1$) or the absence ($m_l=0$) of the π flux in the hole.

For the degenerate ground states on an even-by-odd ($e*o$) lattice (L_x is even and L_y is odd), the situation changes. Because a Z_2 vortex or Z_2 charge has to move even steps to go back to the same plaquette around a torus, we cannot well define a topological closed-string operator of Z_2 vortex or Z_2 charge along e_y direction, of which the loop consists of odd-number plaquettes. Then we can only define topological closed-string operator of Z_2 vortex and Z_2 charge along e_x direction $W(\mathcal{C}_X)$ $[W(\mathcal{C}_X)=W_v(\mathcal{C}_X)=W_c(\mathcal{C}_X)]$ and the corresponding fermionic string operator along e_y direction $W_f(\mathcal{C}_Y)$. Due to the anticommutation relations between $W(\mathcal{C}_X)$ and $W_f(\mathcal{C}_Y)$,

$$\{W(\mathcal{C}_X),W_f(\mathcal{C}_Y)\}=0, \tag{10}$$

we may represent $W(\mathcal{C}_X)$ and $W_f(\mathcal{C}_Y)$ by pseudospin operators τ_1^x and τ_1^z, respectively,

$$W(\mathcal{C}_X) \to \tau_1^x, \quad W_f(\mathcal{C}_Y) \to \tau_1^z. \tag{11}$$

Therefore there are two degenerate ground states $|m_1\rangle$ that are the eigenstates of τ_1^z. In Table II, the pseudospin representation of the topological closed-string operators on $e*o$ lattice are shown.

Similarly, for the degenerate ground states on an odd-by-even ($o*e$) lattice there are also two types of closed-string operators, $W(\mathcal{C}_Y)$ $[W(\mathcal{C}_Y)=W_v(\mathcal{C}_Y)=W_c(\mathcal{C}_Y)]$ and $W_f(\mathcal{C}_X)$, which can be described by pseudospin operators τ_2^x and τ_2^z. Therefore, the two degenerate ground states on an $o*e$ lattice are denoted by $|m_2\rangle$ which are the eigenstates of τ_2^z.

For the degenerate ground states on an odd-by-odd ($o*o$) lattice, since the total lattice number is odd, we cannot well define Z_2 vortex or Z_2 charge globally any more. Instead, we can only define a mixed topological closed-string operator, $W(\mathcal{C}_{XY})=\Pi_{\mathcal{C}}\sigma_i^{a_i}$, where the product $\Pi_{\mathcal{C}}$ is over all the sites on the string along a diagonal loop $\mathcal{C}$ connecting plaquettes. The index $a_i=x$ or y is determined by the position of the plaquettes. Because $W(\mathcal{C}_{XY})$ anticommutes with $W_f(\mathcal{C}_X)$ and $W_f(\mathcal{C}_Y)$,

$$\{W(\mathcal{C}_{XY}),W_f(\mathcal{C}_X)\}=0, \quad \{W(\mathcal{C}_{XY}),W_f(\mathcal{C}_Y)\}=0, \tag{12}$$

we may represent $W(\mathcal{C}_{XY})$ and $W_f(\mathcal{C}_X)$ [or $W_f(\mathcal{C}_Y)$] by pseudospin operators τ^x and τ^z, respectively,

$$W(\mathcal{C}_{XY}) \to \tau^x,$$

$$W_f(\mathcal{C}_X)=W_f(\mathcal{C}_Y) \to \tau^z.$$

It is noted that

TABLE III. Pseudospin representation of the topological closed-string operators on an odd-by-odd lattice.

Pseudospin operators	$\mathcal{C}_X$	$\mathcal{C}_Y$	$\mathcal{C}_{XY}$
Z_2 vortex (Z_2 charge)			τ^x
Fermion	τ^z	τ^z	1

$$W_f(\mathcal{C}_{XY})=W_f(\mathcal{C}_X)W_f(\mathcal{C}_Y)=1. \tag{13}$$

Thus the two degenerate ground states on an $o*o$ lattice $|m\rangle$ are the eigenstates of τ^z. In Table III, the pseudospin representation of the topological closed-string operators on $o*o$ lattice are shown.

As a result, the degeneracy $\mathcal{Q}$ of the ground states of the Wen-plaquette model on lattices with periodic boundary condition (on a torus) is dependent on the lattice numbers: $\mathcal{Q}=4$ on $e*e$ lattice, $\mathcal{Q}=2$ on other cases ($e*o$, $o*e$, and $o*o$ lattices).[2,5,7,10–12]

III. PROPERTIES OF QUASIPARTICLES OF THE WEN-PLAQUETTE MODEL

In this section we study the properties of the quasiparticles of the Wen-plaquette model. In this model, Z_2 vortex is defined as $F_i=-1$ at even subplaquette and Z_2 charge is $F_i=-1$ at odd subplaquette. The energy gaps of Z_2 charge and Z_2 vortex are $2g$. The fermions that are the bound states of a Z_2 charge and a Z_2 vortex on two neighboring plaquettes have an energy gap of $4g$. All quasiparticles in such an exactly solvable model have flat bands. The energy spectrums are $E_v=E_c=2g$ for Z_2 vortex and Z_2 charge, $E_f=4g$ for fermions, respectively. In other words, the quasiparticles cannot move at all. In particular, there exist two types of fermions: the fermions on the vertical links and the fermions on the parallel links.

Under the perturbation

$$H_I=h^x\sum_i \sigma_i^x+h^z\sum_i \sigma_i^z, \tag{14}$$

the quasiparticles (Z_2 vortex, Z_2 charge, and fermion) begin to hop.[11,12,14,17–21] The term $h^x\Sigma_i\sigma_i^x$ drives the Z_2 vortex, Z_2 charge, and fermion hopping along diagonal direction $\hat{e}_x-\hat{e}_y$ [see Fig. 3(a)]. For example, for a Z_2 vortex living at i

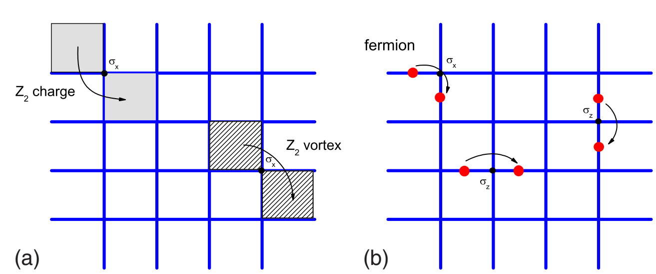

FIG. 3. (Color online) The hoppings of Z_2 vortex, Z2 charge, and fermions. The shadow plaquettes, the striped plaquettes, and the dots on the links represent Z_2 vortices, Z_2 charges, and fermions, respectively.

plaquette $F_i=-1$, when σ_i^x acts on $i+\hat{e}_x$ site, it hops to $i+\hat{e}_x-\hat{e}_y$ plaquette denoted by $F_{i+\hat{e}_x-\hat{e}_y}=-1$,

$$F_i=-1 \rightarrow F_i=+1, \quad F_{i+\hat{e}_x-\hat{e}_y}=+1 \rightarrow F_{i+\hat{e}_x-\hat{e}_y}=-1. \tag{15}$$

A pair of Z_2 vortices at i and $i+\hat{e}_x-\hat{e}_y$ plaquettes can be created or annihilated by the operation of σ_i^x,

$$F_i=+1 \rightarrow F_i=-1, \quad F_{i+\hat{e}_x-\hat{e}_y}=+1 \rightarrow F_{i+\hat{e}_x-\hat{e}_y}=-1. \tag{16}$$

The term $h^z\Sigma_i\sigma_i^z$ drives fermion hopping along $\hat{e}_x$ and $\hat{e}_y$ directions without affecting Z_2 vortex and Z_2 charge: fermions on the vertical links move along vertical directions and fermions on the parallel links move along parallel directions. With the help of the term $h^x\Sigma_i\sigma_i^x$, the two types of fermions are mixed and the fermions may turn round from vertical links to parallel links [see Fig. 3(b)].

A fact is that *the topological closed-string operators can be considered as quantum tunneling processes of virtual quasiparticle moving along the same loops*. Let us take the quantum tunneling process of Z_2 vortex as an example; at first a pair of Z_2 vortices are created. One Z_2 vortex propagates around the torus driven by operators σ_i^x and annihilates with the other Z_2 vortex. Then a string of σ_i^x is left on the tunneling path, which is just the topological closed-string operator $W_v(\mathcal{C})$. Such a process effectively adds a unit of a π flux to one hole of the torus and changes m_l by 1 (from the case of $m_l=1$ to that of $m_l=0$, or vice versa).

IV. MACROSCOPIC QUANTUM TUNNELING EFFECTS OF THE DEGENERATE GROUND STATES

It is known that the degenerate ground states of Z_2 topological orders have the same energy in the thermodynamic limit. The different ground states cannot mix with each other through any local fluctuations. However, in a finite system, the degeneracy of the ground states can be (partially) removed due to quantum tunneling processes, during which virtual quasiparticles move around the torus.[2,6,9,11,12] In general cases, one will get large energy gaps for all quasiparticles and very tiny energy splittings of the degenerate ground states ΔE. Based on such condition, we may ignore high-energy excited states and consider only the degenerate ground states. Thus in the following parts we only focus on the ground states that are a four-level (or two-level) system.

A. High-order degenerate perturbation theory

To solve quantum tunneling problems, people have developed many approaches including the well-known WKB (Wentzel, Kramers and Brillouin) method and the instanton approach lately. However, both above approaches are based on semiclassical approximation and are not available to the MQT of Z_2 topological order. Instead, in this part, we develop a high-order degenerate perturbative approach to calculate the MQT.

The Hamiltonian of the Wen-plaquette model under the external field has a form as

$$\hat{H}=\hat{H}_0+\hat{H}_I \tag{17}$$

in which $\hat{H}_0=-g\Sigma_i\sigma_i^x\sigma_{i+\hat{e}_x}^y\sigma_{i+\hat{e}_x+\hat{e}_y}^x\sigma_{i+\hat{e}_y}^y$ is the unperturbation term and $\hat{H}_I=h^x\Sigma_i\sigma_i^x+h^z\Sigma_i\sigma_i^z$ is the small perturbation one. For simplicity, we consider the quantum tunneling process between two degenerate ground states $|m\rangle$ and $|n\rangle$,

$$|m\rangle \Leftrightarrow |n\rangle. \tag{18}$$

According to the Gell-Mann-Low theory, we define a transformation operator $\hat{U}_I(0,-\infty)$ as

$$U_I(0,-\infty)=T\exp\left(-i\int_{-\infty}^0 \hat{H}_I'(t')dt'\right), \tag{19}$$

where

$$\hat{H}_I'(t)=e^{i\hat{H}_0t}\hat{H}_Ie^{-i\hat{H}_0t}. \tag{20}$$

Here T is a time-ordering operator and $\hbar=1$. Then the transformation operator $\hat{U}_I(0,-\infty)$ in Eq. (19) can be written as

$$\hat{U}_I(0,-\infty)|m\rangle=\sum_{j=0}^{\infty}\hat{U}_I^{(j)}(0,-\infty)|m\rangle, \tag{21}$$

where

$$\hat{U}_I^{(0)}(0,-\infty)|m\rangle=|m\rangle,$$

$$\hat{U}_I^{(1)}(0,-\infty)|m\rangle=-i\int_{-\infty}^0\hat{H}_I'(t)dt|m\rangle=\frac{1}{E_0-\hat{H}_0}\hat{H}_I|m\rangle,$$

$$\hat{U}_I^{(2)}(0,-\infty)|m\rangle=-i\int_{-\infty}^0\hat{H}_I'(t)\hat{U}_I^{(1)}(0,-\infty)dt|m\rangle$$

$$=\frac{1}{E_0-\hat{H}_0}\hat{H}_I\frac{1}{E_0-\hat{H}_0}\hat{H}_I|m\rangle,$$

$$\hat{U}_I^{(j\neq 0)}(0,-\infty)|m\rangle=\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)^j|m\rangle. \tag{22}$$

The element of the transformation matrix from the state $|m\rangle$ to $|n\rangle$ becomes

$$\langle n|\hat{U}_I(0,-\infty)|m\rangle$$

and the corresponding energy is obtained as

$$E=\langle n|\hat{H}\hat{U}_I(0,-\infty)|m\rangle=E_0+\delta E, \tag{23}$$

where E_0 is the eigenvalue of the Hamiltonian $\hat{H}_0$ of $|m\rangle$.

For the tunneling process from $|m\rangle$ to $|n\rangle$, a quasiparticle will move around the torus that leads to topological closed-string operator behind. So in the summation of j, the dominating term is labeled by $j=L-1$. L is the length of the loop of a topological string operator $W_\upsilon(\mathcal{C}_\Lambda)$, where $\upsilon=v$, c, or f and $\Lambda=X$, Y, or XY. Then considering the tunneling process corresponding to $W_v(\mathcal{C}_\Lambda)$, we obtain the perturbative energy as

$$\delta E = \langle n|\hat{H}_I\hat{U}_I(0,-\infty)|m\rangle = \langle n|\hat{H}_I\sum_{j=0}^{\infty}\hat{U}_I^{(j)}(0,-\infty)|m\rangle$$
$$= \langle n|\hat{H}_I\hat{U}_I^{(L-1)}(0,-\infty)|m\rangle. \tag{24}$$

Now it is noting that the operator $\hat{H}_I\hat{U}_I^{(L-1)}(0,-\infty)$ is proportional to a topological string operator $W_v(\mathcal{C}_\Lambda)$. In general, the perturbative energy δE is proportional to

$$N\frac{(t_{\text{eff}})^L}{(\delta\epsilon)^{L-1}} \tag{25}$$

with the lattice number N and the tunneling path length L. Here $\delta\epsilon$ and t_{eff} are the energy gap and the hopping amplitude of the quasiparticle, respectively.

Considering all tunneling processes, we may denote the ground-state energies as a four-by-four matrix (for the four degenerate ground states on $e*e$ lattice) or a two-by-two matrix (for the two degenerate ground states on $e*o$, $o*e$, and $o*o$ lattices),

$$\delta E = \sum_{m,n}\langle n|\hat{H}_I\hat{U}_I^{(L-1)}(0,-\infty)|m\rangle. \tag{26}$$

Finally we can diagonalize the four-by-four or two-by-two matrices and obtain the energy splitting.

In the following parts, by using this high-order perturbative approach we study the tunneling splitting between topological degenerate ground states. In the sense of "perturbative," such approach can only be applied to the cases under a small external field. When the ratio $\frac{h^x}{g}$ (or $\frac{h^z}{g}$) is large, our approach is not reliable. Near the phase boundary of the topological ordered state, the energy gaps of one or more quasiparticle disappear, $\delta\epsilon\to 0$. According to Eq. (25), one gets a diverge perturbative energy $\delta E\sim N\frac{(t_{\text{eff}})^L}{(\delta\epsilon)^{L-1}}\to\infty$. So by the approach we cannot study the quantum tunneling effect near the quantum phase transition out of the topological order.[19,21–24]

B. Macroscopic quantum tunneling effect of the degenerate ground states on $o*o$ lattice

First, we study the MQT of the two degenerate ground states on an $L_x\times L_y$ (L_x and L_y are odd numbers and $L_x \geq L_y$) lattice. For simplicity, we use $|\uparrow\rangle$ and $|\downarrow\rangle$ to describe the two degenerate ground states $|m=0\rangle$ and $|m=1\rangle$, respectively. Therefore the two ground states are mapped onto quantum states of pseudospin $\hat{\tau}$. Under the perturbation, $\hat{H}_I = h^x\Sigma_i\sigma_i^x + h^z\Sigma_i\sigma_i^z$, two types of quantum tunneling processes dominate—the one that Z_2 vortex (or Z_2 charge) propagates around the torus along diagonal direction and the other that fermion propagates around the torus along e_y direction.

For the first tunneling process, a virtual Z_2 vortex (or Z_2 charge) will run around the torus as long as a path with length L_0 that is equal to $\frac{L_xL_y}{\xi}$. Here ξ is the maximum common divisor for L_x and L_y. For example, on a 3×3 lattice, we get $L_0=\frac{3\times 3}{3}=3$; on a 3×5 lattice, we get $L_0=\frac{5\times 3}{1}=15$.

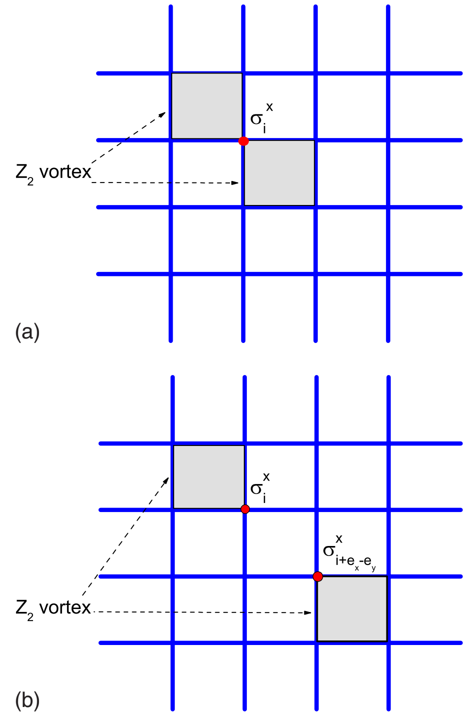

FIG. 4. (Color online) Generation and hopping of Z_2 vortex. The shadow plaquettes represent Z_2 vortices.

From Eq. (26), one may obtain the energy splitting ΔE of the two ground states as

$$\delta E = U_I^{(L)} = \langle\uparrow|\hat{H}_I\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)^{L_0-1}|\downarrow\rangle. \tag{27}$$

Due to the translation invariance, to calculate $(\frac{1}{E_0-\hat{H}_0}\hat{H}_I)|\downarrow\rangle = (\frac{h^x}{E_0-\hat{H}_0}\Sigma_i\sigma_i^x)|\downarrow\rangle$, we can choose site i as the starting point of the tunneling process and get

$$\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)|\downarrow\rangle \to L_xL_y\left(\frac{h^x}{E_0-\hat{H}_0}\sigma_i^x\right)|\downarrow\rangle$$
$$= L_xL_y\left(\frac{h^x}{E_0-\hat{H}_0}\right)|\Psi_i\rangle, \tag{28}$$

where $|\Psi_i\rangle$ is the excited state of two Z_2 vortices (or Z_2 charges) at plaquettes $i-e_y$ and $i-e_x$ with an energy E_0+4g [see Fig. 4(a)]. From $\hat{H}_0|\Psi_i\rangle=(E_0+4g)|\Psi_i\rangle$, we have

$$\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)|\downarrow\rangle = L_xL_y\left(\frac{h^x}{-4g}\right)|\Psi_i\rangle.$$

In second step, one Z_2 vortex (or Z_2 charge) moves one step, we get

$$\left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)^2|\downarrow\rangle = \left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)L_xL_y\left(\frac{h^x}{-4g}\right)|\Psi_i\rangle = L_xL_y\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)|\Psi_i\rangle = L_xL_y\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{E_0-\hat{H}_0}\sigma^x_{i+e_x-e_y}\right)|\Psi_i\rangle = L_xL_y\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{-4g}\right)|\Psi_i'\rangle, \quad (29)$$

where $|\Psi_i'\rangle$ is the excited state of two Z_2 vortices (or Z_2 charges) at plaquettes $i+e_x-2e_y$ and $i-e_x$. See Fig. 4(b).

In third step, one Z_2 vortex (or Z_2 charge) moves two steps, we get

$$\left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)^3|\downarrow\rangle = \left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)^2L_xL_y\left(\frac{h^x}{-4g}\right)|\Psi_i\rangle = \left(\frac{h^x}{E_0-\hat{H}_0}\sum_i \sigma_i^x\right)L_xL_y\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{-4g}\right)\times|\Psi_i'\rangle = L_xL_y\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{-4g}\right)\left(\frac{h^x}{-4g}\right)\times|\Psi_i''\rangle, \quad (30)$$

where $|\Psi_i''\rangle$ is the excited state of two Z_2 vortices (or Z_2 charges) at plaquettes $i+2e_x-3e_y$ and $i-e_x$.

Then step by step, one Z_2 vortex (or Z_2 charge) moves around the torus. When the Z_2 vortex (or Z_2 charge) goes back to its starting point and annihilates with the other, the original quantum state $|\downarrow\rangle$ changes into $|\uparrow\rangle$. Finally we get the energy splitting

$$\Delta = 2\delta E = 2U_I^{(L)} = 2\langle\uparrow|\hat{H}_I\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)^{L_0-1}|\downarrow\rangle = 2\times L_xL_y\frac{(h^x)^{L_0}}{(-4g)^{L_0-1}} = 8L_xL_yg\left(\frac{h^x}{4g}\right)^{L_0}. \quad (31)$$

It is noted that L_0-1 is an even number.

Because the quantum tunneling process of Z_2 vortex (or Z_2 charge) acts on the quantum states $\binom{|\uparrow\rangle}{|\downarrow\rangle}$ as τ^x

$$\begin{pmatrix}|\downarrow\rangle\\|\uparrow\rangle\end{pmatrix} = \tau^x\begin{pmatrix}|\uparrow\rangle\\|\downarrow\rangle\end{pmatrix}, \quad (32)$$

we obtain the effective pseudospin Hamiltonian due to the contribution of Z_2 vortex (or Z_2 charges) as

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\Delta}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = J_x\tau^x \quad (33)$$

where $J_x=\Delta/2$.[11,12]

For the second tunneling process, a virtual fermion will move around the torus along direction $\hat{e}_y$ with length L_y (it is noted that due to $L_x \geq L_y$, the length of tunneling path along

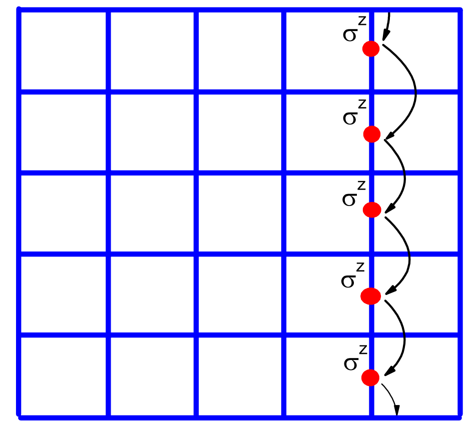

FIG. 5. (Color online) Tunneling path of virtual fermion along $\hat{e}_y$ direction on an 5×5 lattice (The dots on the links denote the fermions).

$\hat{e}_x$ direction is longer). See Fig. 5. Such a tunneling process changes the quantum states $\binom{|\uparrow\rangle}{|\downarrow\rangle}$ turn into $\binom{|\uparrow\rangle}{-|\downarrow\rangle}=\tau^z\binom{|\uparrow\rangle}{|\downarrow\rangle}$. The extra sign of the state $|\downarrow\rangle$ comes from the presence of π flux of fermionic quasiparticles through the holes of the torus. From Eq. (26), we can get the energy shift of the state $|\downarrow\rangle$ as

$$\delta E = \sum_{j=0}^{\infty}\langle\downarrow|\hat{H}_I\left(\frac{1}{E_0-\hat{H}_0}\hat{H}_I\right)^j|\downarrow\rangle = L_xL_y\frac{(h^z)^{L_y}}{(8g)^{L_y-1}} \quad (34)$$

with an even number L_y-1. Through the same approach, we get the energy shift ΔE of $|\downarrow\rangle$ is equal to $-L_xL_y\frac{(h^z)^{L_y}}{(8g)^{L_y-1}}$. Then an energy difference ε of the two ground states is obtained as

$$\varepsilon = 2\delta E = 16L_xL_yg\left(\frac{h^z}{8g}\right)^{L_y}. \quad (35)$$

Finally the two-level quantum system of the two degenerate ground states on an $o*o$ lattice can be described by a simple effective pseudospin Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\Delta}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) + \frac{\varepsilon}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = J_x\tau^x + J_z\tau^z, \quad (36)$$

where $J_x=\Delta/2$ and $J_z=\varepsilon/2$. By diagonalizing the effective Hamiltonian matrix, we can get the eigenvalues of the two ground states

$$E_{\pm} = \pm\sqrt{\left(\frac{\Delta}{2}\right)^2+\left(\frac{\varepsilon}{2}\right)^2}. \quad (37)$$

The total-energy splitting becomes

$$\Delta E = E_+ - E_- = 2\sqrt{\left(\frac{\Delta}{2}\right)^2+\left(\frac{\varepsilon}{2}\right)^2}. \quad (38)$$

For the Wen-plaquette model under external field along x direction, the total-energy splitting ΔE is reduced into $\Delta = 8L_xL_yg(\frac{h^x}{4g})^{L_0}$. On the other hand, for the Wen-plaquette model under external field along z direction, the total-energy splitting ΔE is $\varepsilon=16L_xL_yg(\frac{h^z}{8g})^{L_y}$.

C. Macroscopic quantum tunneling effect of the degenerate ground states on *e*∗*o* lattice

Second, we study the MQT of the two degenerate ground states on an $L_x \times L_y$ (L_x is an even number and L_y is an odd number) lattice.[25] Now we map the twofold-degenerate ground states $|m_1=0\rangle$ and $|m_1=1\rangle$ onto quantum states of the pseudospin $\hat{\tau}_1$ as $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$, respectively. Under the perturbation, $\hat{H}_I = h^x \Sigma_i \sigma_i^x + h^z \Sigma_i \sigma_i^z$, there are two types of quantum tunneling processes—virtual Z_2 vortex (or Z_2 charge) propagating along $\hat{e}_x - \hat{e}_y$ directions around the torus and virtual fermion propagating along $\hat{e}_y$ direction around the torus.

For the virtual Z_2 vortex (or Z_2 charge) propagating along $\hat{e}_x - \hat{e}_y$ directions around the torus, the energy splitting Δ can be obtained by the high-order degenerate-state perturbation theory as

$$\Delta = 2\langle\uparrow|_1 \hat{H}_I \left(\frac{1}{E_0 - \hat{H}_0}\hat{H}_I\right)^{L_0-1} |\downarrow\rangle_1 = 2L_x L_y \frac{(h^x)^{L_0}}{(-4g)^{L_0-1}}. \tag{39}$$

Because the quantum tunneling process of Z_2 vortex (or Z_2 charge) plays a role of τ_1^x on the quantum states $\binom{|\uparrow\rangle_1}{|\downarrow\rangle_1}$ as $\binom{|\downarrow\rangle_1}{|\uparrow\rangle_1} = \tau_1^x \binom{|\uparrow\rangle_1}{|\downarrow\rangle_1}$, we obtain the effective pseudospin Hamiltonian due to the contribution of Z_2 vortex (or Z_2 charge) as

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\Delta}{2}(|\uparrow\rangle_1\langle\downarrow|_1 + |\downarrow\rangle_1\langle\uparrow|_1) = J_x \tau_1^x, \tag{40}$$

where $J_x = \Delta/2$.

For the tunneling process of fermion propagating around the torus along direction $\hat{e}_y$, we obtain the energy difference ε of the two ground states as

$$\varepsilon = 2\Delta E = 16 L_x L_y g \left(\frac{h^z}{8g}\right)^{L_y}. \tag{41}$$

The length of the tunneling path is L_y which is an odd number. Such tunneling process plays a role of τ_1^z.

Finally the two-level quantum system of the two degenerate ground states on an *e*∗*o* lattice can be described by

$$\hat{\mathcal{H}}_{\text{eff}} = J_x \tau_1^x + J_z \tau_1^z, \tag{42}$$

where $J_x = \Delta/2$ and $J_z = \varepsilon/2$. The total-energy splitting now becomes

$$\Delta E = E_+ - E_- = 2\sqrt{\left(\frac{\Delta}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2}. \tag{43}$$

In Figs. 6 and 7, we plot the numerical results from the exact diagonalization technique of the Wen-plaquette model on different *o*∗*o* and *e*∗*o* lattices. Table IV shows the tunneling lengths L_0 from the numerical results (the numbers in the brackets are the theoretical predictions), which indicate that our theoretical results are consistent with the numerical results from exact diagonalization approach.

D. Macroscopic quantum tunneling effect of the degenerate ground states on *e*∗*e* lattice

Third, we study the MQT of the four degenerate ground states on an $L_x \times L_y$ (L_x and L_y are even numbers with $L_x \geq L_y$) lattice. We denote the four degenerate ground states $|m_1, m_2\rangle = |0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$, and $|1,1\rangle$ by the quantum states of pseudospin $\hat{\tau}_1$ and $\hat{\tau}_2$. Under the perturbation, $\hat{H}_I = h^x \Sigma_i \sigma_i^x + h^z \Sigma_i \sigma_i^z$, there are five types of quantum tunneling processes—virtual Z_2 vortex propagating along $\hat{e}_x - \hat{e}_y$ direction around the torus, Z_2 charge propagating along $\hat{e}_x - \hat{e}_y$ direction around the torus, and virtual fermion propagating along $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_x - \hat{e}_y$ direction around the torus, respectively. We will calculate the ground-state energy splitting from the degenerate perturbation approach one by one.

In the first step we study the quantum tunneling process of Z_2 vortex propagating along $\hat{e}_x - \hat{e}_y$ direction around the torus. After such tunneling process, the quantum states

$$\begin{pmatrix} |0,0\rangle \\ |1,0\rangle \\ |0,1\rangle \\ |1,1\rangle \end{pmatrix}$$

turn into

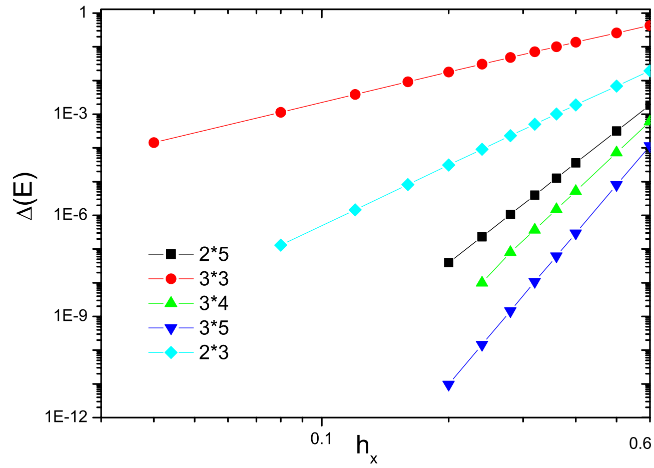

FIG. 6. (Color online) The energy splitting between the two degenerate ground states of the Wen-plaquette model in an external field along x direction ($g=1$). Here $N*M$ denotes a $N \times M$ lattice.

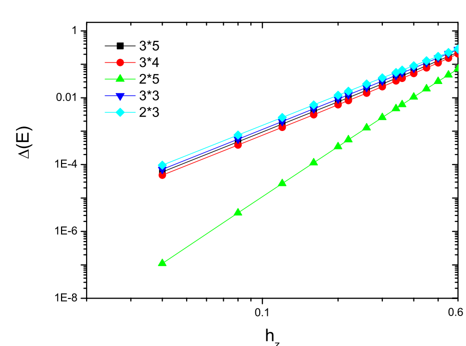

FIG. 7. (Color online) The energy splitting between the two degenerate ground states of the Wen-plaquette model in an external field along z direction ($g=1$). Here $N*M$ denotes a $N \times M$ lattice.

TABLE IV. The tunneling lengths L_0 from the numerical results (the numbers in the brackets are the theoretical predictions). h_x means the external field along x direction and h_z means the external field along z direction. Here $N*M$ denotes a $N\times M$ lattice.

L_0	3*3	2*5	3*4	3*5
h_x	2.98312(3)	9.84653(10)	12.10754(12)	15.01707(15)
h_z	3.06994(3)	4.83737 (5)	3.03164 (3)	3.01557 (3)

$$\begin{pmatrix}|0,0\rangle\\|1,0\rangle\\|0,1\rangle\\|1,1\rangle\end{pmatrix}\rightarrow\begin{pmatrix}|1,1\rangle\\|0,1\rangle\\|1,0\rangle\\|0,0\rangle\end{pmatrix}=\tau_1^x\otimes\tau_2^x\begin{pmatrix}|0,0\rangle\\|1,0\rangle\\|0,1\rangle\\|1,1\rangle\end{pmatrix}. \quad (44)$$

Thus we may use the pseudospin operator $\tau_1^x\otimes\tau_2^x$ to denote the tunneling process. The effective pseudospin Hamiltonian due to the contribution of Z_2 vortex is obtained as

$$\hat{\mathcal{H}}_{\rm eff}=J_{xx}\tau_1^x\otimes\tau_2^x, \quad (45)$$

where $J_{xx}=L_xL_y\frac{(h^x)^{L_0}}{(-4g)^{L_0-1}}$. Similar to the results in above section, the length L_0 of the tunneling path is equal to $\frac{L_xL_y}{\xi}$, where ξ is the maximum common divisor for L_x and L_y.

In the second step we study the quantum tunneling process of Z_2 charge propagating along $\hat{e}_x-\hat{e}_y$ directions around the torus. We may use the pseudospin operator $\tau_1^y\otimes\tau_2^y$ to denote this tunneling process, of which the effective pseudospin Hamiltonian is obtained as

$$\hat{\mathcal{H}}_{\rm eff}=J_{yy}\tau_1^y\otimes\tau_2^y, \quad (46)$$

where $J_{yy}=L_xL_y\frac{(h^x)^{L_0}}{(-4g)^{L_0-1}}$ and $L_0=\frac{L_xL_y}{\xi}$.

In the third step we study the quantum tunneling process of fermions propagating along $\hat{e}_x$ and $\hat{e}_y$ directions around the torus, of which the pseudospin operators correspond to $\mathbf{1}\otimes\tau_2^z$ and $\tau_1^z\otimes\mathbf{1}$, respectively. Then the effective pseudospin Hamiltonian due to the contribution of the two quantum tunneling processes is obtained as

$$\hat{\mathcal{H}}_{\rm eff}=\tilde{h}_1^z(\tau_1^z\otimes\mathbf{1})+\tilde{h}_2^z(\mathbf{1}\otimes\tau_2^z), \quad (47)$$

where $\tilde{h}_1^z=-8L_xL_yg(\frac{h^z}{8g})^{L_x}$ and $\tilde{h}_2^z=-8L_xL_yg(\frac{h^z}{8g})^{L_y}$.

In the last step we study the quantum tunneling process of fermions propagating along $\hat{e}_x-\hat{e}_y$ direction around the torus, of which the pseudospin operator corresponds to $\tau_1^z\otimes\tau_2^z$, respectively. Now there are a lot of tunneling paths with same length $2L_0$. Different tunneling paths can be labeled by the positions of corners, at which the fermions make a turn round from vertical links to parallel links (or parallel links to vertical links). For a path with $2k$ corners (k is an positive integer number), the topological closed-string operator can be written as

$$W_f(\mathcal{C}_{XY})=\sigma_i^z\sigma_{i+1}^z\ldots\sigma_{j-1}^z\sigma_j^x\sigma_{j+1}^z\ldots\sigma_{2L_0-2}^x\sigma_{2L_0-1}^z\sigma_{2L_0}^z \quad (48)$$

with the site $i=(i_x,i_y)$ and a neighboring site $i+1$. See Fig. 8. Along the closed loops, each operator σ_j^x corresponds to a corner. Therefore, the number of paths with $2k$ corners that is equal to the power of h^x in ΔE is obtained as

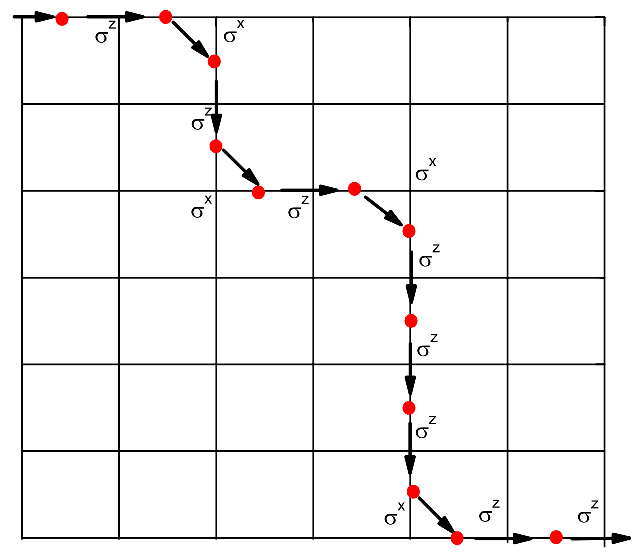

FIG. 8. (Color online) Tunneling path of virtual fermions on an 6×6 lattice with four corners (the dots on the links denote fermions).

$$C_{L_0-1}^k=\frac{(L_0-1)!}{k!\,(L_0-k-1)!}. \quad (49)$$

It is noted that for any path, there are at least two corners. Then after considering the tunneling processes of all possible paths, the matrix element of $\tau_1^z\otimes\tau_2^z$ is obtained as

$$\begin{aligned}J_{zz}=\varepsilon&=L_xL_yC_{L_0-1}^1\frac{(2h^x)^2(h^z)^{2L_0-2}}{(-8g)^{2L_0-1}}\\&+L_xL_yC_{L_0-1}^2\frac{(2h^x)^4(h^z)^{2L_0-4}}{(-8g)^{2L_0-1}}+\ldots\\&+L_xL_yC_{L_0-1}^k\frac{(2h^x)^{2k}(h^z)^{2L_0-2k}}{(-8g)^{2L_0-1}}+\ldots+L_xL_y\frac{(2h^x)^{2L_0}}{(-8g)^{2L_0-1}}\\&=-L_xL_y\frac{[(2h^x)^2+(h^z)^2]^{L_0}-(h^z)^{2L_0}}{(8g)^{2L_0-1}}. \quad (50)\end{aligned}$$

Finally we derive an effective pseudospin Hamiltonian of the four ground states as

$$\begin{aligned}\hat{\mathcal{H}}_{\rm eff}\simeq{}&J_{xx}(\tau_1^x\otimes\tau_2^x)+J_{yy}(\tau_1^y\otimes\tau_2^y)+J_{zz}(\tau_1^z\otimes\tau_2^z)+\tilde{h}_1^z(\tau_1^z\otimes\mathbf{1})\\&+\tilde{h}_2^z(\mathbf{1}\otimes\tau_2^z) \quad (51)\end{aligned}$$

which is equal to

$$\hat{\mathcal{H}}_{\text{eff}} = \begin{pmatrix} J_{zz} + \widetilde{h_1^z} + \widetilde{h_2^z} & 0 & 0 & J_{xx} - J_{yy} \\ 0 & -J_{zz} + \widetilde{h_1^z} - \widetilde{h_2^z} & J_{xx} + J_{yy} & 0 \\ 0 & J_{xx} + J_{yy} & -J_{zz} - \widetilde{h_1^z} + \widetilde{h_2^z} & 0 \\ J_{xx} - J_{yy} & 0 & 0 & J_{zz} - \widetilde{h_1^z} - \widetilde{h_2^z} \end{pmatrix}. \tag{52}$$

The coefficients of $\hat{\mathcal{H}}_{\text{eff}}$ are given by

$$J_{xx} = J_{yy} = L_x L_y \frac{(h^x)^{L_0}}{(-4g)^{L_0-1}},$$

$$J_{zz} = -L_x L_y \frac{[(2h^x)^2 + (h^z)^2]^{L_0} - (h^z)^{2L_0}}{(8g)^{2L_0-1}},$$

$$\widetilde{h}_1^z = -8L_x L_y g\left(\frac{h^z}{8g}\right)^{L_x}, \quad \widetilde{h}_2^z = -8L_x L_y g\left(\frac{h^z}{8g}\right)^{L_y}. \tag{53}$$

By diagonalizing the effective Hamiltonian, we get the energies of the ground states as

$$E_1 = J_{zz} - \sqrt{(\widetilde{h}_1^z - \widetilde{h}_2^z)^2 + 4J_{xx}^2},$$

$$E_2 = J_{zz} + \sqrt{(\widetilde{h}_1^z - \widetilde{h}_2^z)^2 + 4J_{xx}^2},$$

$$E_3 = J_{zz} + \widetilde{h}_1^z + \widetilde{h}_2^z,$$

$$E_4 = J_{zz} - \widetilde{h}_1^z - \widetilde{h}_2^z. \tag{54}$$

Because the parameter J_{zz} is always much smaller than others as $|J_{zz}| \ll |J_{xx}|$, $|J_{yy}|$, $|\widetilde{h}_1^z|$, and $|\widetilde{h}_2^z|$, we may simplify $\hat{\mathcal{H}}_{\text{eff}}$ as[11]

$$\hat{\mathcal{H}}_{\text{eff}} \simeq J_{xx}(\tau_1^x \otimes \tau_2^x) + J_{yy}(\tau_1^y \otimes \tau_2^y) + \widetilde{h}_1^z(\tau_1^z \otimes \mathbf{1}) + \widetilde{h}_2^z(\mathbf{1} \otimes \tau_2^z) \tag{55}$$

and obtain the energies as

$$E_1 \simeq -\sqrt{(\widetilde{h}_1^z - \widetilde{h}_2^z)^2 + 4J_{xx}^2},$$

$$E_2 \simeq \sqrt{(\widetilde{h}_1^z - \widetilde{h}_2^z)^2 + 4J_{xx}^2},$$

$$E_3 \simeq \widetilde{h}_1^z + \widetilde{h}_2^z,$$

$$E_4 \simeq -\widetilde{h}_1^z - \widetilde{h}_2^z. \tag{56}$$

As the external field increases ($h^x \neq 0$ and $h^z \neq 0$), the single energy level of the initial four degenerate ground states split into four energy levels.

If we apply the external field along z direction, the four energy levels are

$$E_1 \simeq -\widetilde{h}_1^z + \widetilde{h}_2^z, \quad E_2 \simeq \widetilde{h}_1^z - \widetilde{h}_2^z,$$

$$E_3 \simeq \widetilde{h}_1^z + \widetilde{h}_2^z, \quad E_4 \simeq -\widetilde{h}_1^z - \widetilde{h}_2^z, \tag{57}$$

where $\widetilde{h}_1^z = -8L_xL_yg(\frac{h^z}{8g})^{L_x}$ and $\widetilde{h}_2^z = -8L_xL_yg(\frac{h^z}{8g})^{L_y}$. In the anisotropic limit, $L_x \gg L_y$, we have $|\widetilde{h}_1^z| \ll |\widetilde{h}_2^z|$. In this case, the initial four degenerate ground states split into two groups, $E_1 \simeq \widetilde{h}_2^z$, $E_2 \simeq -\widetilde{h}_2^z$, $E_3 \simeq \widetilde{h}_2^z$, and $E_4 \simeq -\widetilde{h}_2^z$. In each group, there are two energy levels, of which the energy splitting $E_1 - E_3 = -2\widetilde{h}_1^z$ is very tiny. In contrast, the energy "gap" between the two groups $E_1 - E_2 = 2\widetilde{h}_2^z$ is larger. One can see the energy levels of the Wen-plaquette model in external field along z direction on 2×6 lattice ($g=1$) in Fig. 9. In the isotropic case, $L_x = L_y$, we have $\widetilde{h}_1^z = \widetilde{h}_2^z$. In this case, the initial four degenerate ground states split into $E_1 = E_2 = 0$, $E_3 = 2\widetilde{h}_1^z$, and $E_4 = -2\widetilde{h}_1^z$. One can see the energy levels of the Wen-plaquette model in external field along z direction on 4×4 lattice ($g=1$) in Fig. 10.

On the other hand, if we apply the external field along x direction, the four energy levels become

$$E_1 \simeq -2J_{xx}, \quad E_2 \simeq 2J_{xx},$$

$$E_3 = E_4 = J_{zz} \simeq 0, \tag{58}$$

where $J_{xx} = L_xL_y\frac{(h^x)^{L_0}}{(-4g)^{L_0-1}}$ and $J_{zz} = -8L_xL_yg(\frac{h^x}{8g})^{2L_0}$. Now the initial four degenerate ground states split into three energy levels. One can see the energy levels of the Wen-plaquette

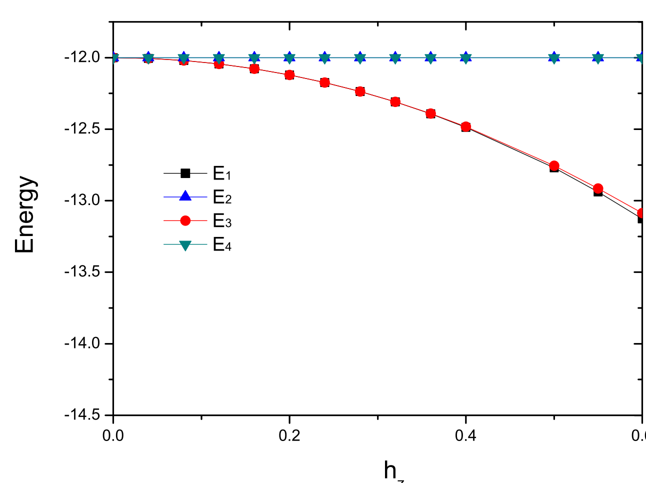

FIG. 9. (Color online) The ground-state energies of the Wen-plaquette model in an external field along z direction on 2×6 lattice ($g=1$).

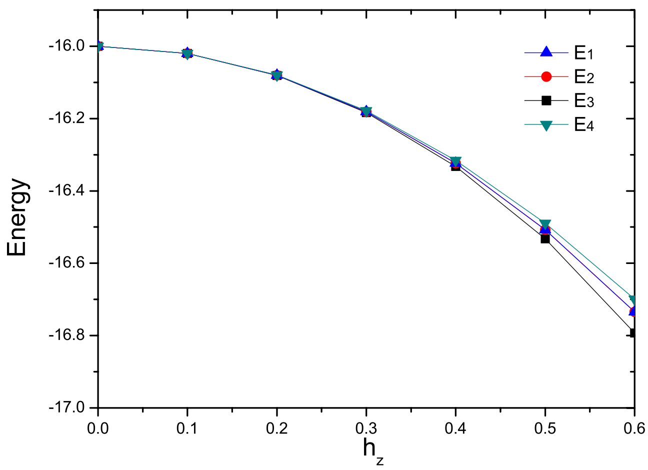

FIG. 10. (Color online) The ground-state energies of the Wen-plaquette model in an external field along z direction on 4×4 lattice $(g=1)$.

model in an external field along x direction on 4×4 lattice $(g=1)$ in Fig. 11.

In addition, one may consider the MQT under a more general perturbation

$$\hat{H}_I = h^x \sum_i \sigma_i^x + h^y \sum_i \sigma_i^y + h^z \sum_i \sigma_i^z . \tag{59}$$

For an external field of $h^x \neq 0$, $h^y \neq 0$, and $h^z \neq 0$, all quasiparticles (Z_2 vortex, Z_2 charge, and fermion) can move along $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_x \pm \hat{e}_y$ directions freely. Therefore, to calculate the MQT of the degenerate ground states on an $e*e$ lattice, all the nine types of quantum tunneling processes should be considered. The corresponding effective pseudospin Hamiltonian of the four ground states turns into

$$\begin{aligned}\mathcal{H}_{\text{eff}} = {} & J_{xx}(\tau_1^x \otimes \tau_2^x) + J_{yy}(\tau_1^y \otimes \tau_2^y) + J_{zz}(\tau_1^z \otimes \tau_2^z) + J_{zx}(\tau_1^z \otimes \tau_2^x) \\ & + J_{xz}(\tau_1^x \otimes \tau_2^z) + \tilde{h}_1^x(\tau_1^x \otimes \mathbf{1}) + \tilde{h}_2^x(\mathbf{1} \otimes \tau_2^x) + \tilde{h}_1^z(\tau_1^z \otimes \mathbf{1}) \\ & + \tilde{h}_2^z(\mathbf{1} \otimes \tau_2^z),\end{aligned} \tag{60}$$

where J_{xx}, J_{yy}, J_{zz}, J_{zx}, J_{xz}, $\tilde{h}_1^x$, $\tilde{h}_2^x$, $\tilde{h}_1^z$, and $\tilde{h}_2^z$ are determined by the energy splitting of the degenerate ground states from the nine tunneling processes.[11,12] This issue [the MQT of Eq. (59)] will be studied elsewhere.

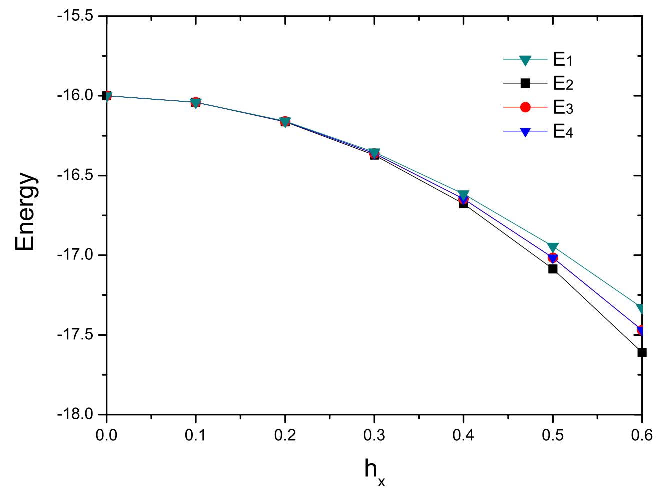

FIG. 11. (Color online) The ground-state energies of the Wen-plaquette model in an external field along x direction on 4×4 lattice $(g=1)$.

V. CONCLUSION

In this paper, we study MQT effect of Z_2 topological order in the Wen-Plaquette model that is characterized by the quantum tunneling processes of different virtual quasiparticles moving around the torus. By focusing on the degenerate ground states, we get their effective pseudospin models. The coefficients of these effective pseudospin models are obtained by a high-order degenerate perturbation approach. With the help of the effective pseudospin models, the energies of the ground states are calculated and the results are consistent with those obtained from exact diagonalization numerical technique.

In the future, the approach will be applied onto the MQTs of Z_2 topological order in other models such as the Kitaev toric-code mode and the Kitaev model on honeycomb lattice. By learning the nature of the MQT of Z_2 topological orders in different models, one may know how to manipulate the degenerate ground states by controlling the external field and then do topological quantum computation within the degenerate ground states.[11,12]

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[25] The MQT on $o*e$ lattice is identical to that on $e*o$ lattice, except that the two degenerate states are denoted by $|0,0\rangle$ and $|0,1\rangle$ rather than $|0,0\rangle$ and $|1,0\rangle$.